

# ON AN ACTION OF THE BRAID GROUP $B_{2g+2}$ ON THE FREE GROUP $F_{2g}$

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ABSTRACT. We construct an action of the braid group  $B_{2g+2}$  on the free group  $F_{2g}$  extending an action of  $B_4$  on  $F_2$  introduced earlier by Reutenauer and the author. Our action induces a homomorphism from  $B_{2g+2}$  into the symplectic modular group  $\mathrm{Sp}_{2g}(\mathbb{Z})$ . In the special case  $g = 2$  we show that the latter homomorphism is surjective and determine its kernel, thus obtaining a braid-type presentation of  $\mathrm{Sp}_4(\mathbb{Z})$ .

## 1. INTRODUCTION

In [9] Christophe Reutenauer and the present author considered the automorphisms  $G, D, \tilde{G}, \tilde{D}$  of the free group  $F_2$  on two generators  $a$  and  $b$  defined by

$$(1.1) \quad \begin{aligned} G : (a, b) &\mapsto (a, ab), & D : (a, b) &\mapsto (ba, b), \\ \tilde{G} : (a, b) &\mapsto (a, ba), & \tilde{D} : (a, b) &\mapsto (ab, b), \end{aligned}$$

(see also [11, Sect. 2.2.2]). Their images under the natural surjection (the abelianization map)  $\pi : \mathrm{Aut}(F_2) \rightarrow \mathrm{Aut}(\mathbb{Z}^2) = \mathrm{GL}_2(\mathbb{Z})$  are the matrices

$$(1.2) \quad \pi(G) = \pi(\tilde{G}) = A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \pi(D) = \pi(\tilde{D}) = B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The matrices  $A$  and  $B$  generate the subgroup  $\mathrm{SL}_2(\mathbb{Z})$  and satisfy the braid relation

$$AB^{-1}A = B^{-1}AB^{-1}.$$

In [9, Lemma 2.1] we observed that  $G, D, \tilde{G}, \tilde{D}$  satisfy similar braid relations in the automorphism group  $\mathrm{Aut}(F_2)$ , namely

$$\begin{aligned} GD^{-1}G &= D^{-1}GD^{-1}, & \tilde{G}D^{-1}\tilde{G} &= D^{-1}\tilde{G}D^{-1} \\ \tilde{G}\tilde{D}^{-1}\tilde{G} &= \tilde{D}^{-1}\tilde{G}\tilde{D}^{-1}, & G\tilde{D}^{-1}G &= \tilde{D}^{-1}G\tilde{D}^{-1}, \end{aligned}$$

together with the commutation relations

$$G\tilde{G} = \tilde{G}G \quad \text{and} \quad D\tilde{D} = \tilde{D}D.$$

These relations allowed us to define a group homomorphism  $f$  from the braid group  $B_4$  on four strands to  $\mathrm{Aut}(F_2)$  by

$$(1.3) \quad f(\sigma_1) = G, \quad f(\sigma_2) = D^{-1}, \quad f(\sigma_3) = \tilde{G},$$

where  $\sigma_1, \sigma_2, \sigma_3$  are the standard generators of  $B_4$ . In [9] we proved that the image  $f(B_4)$  of  $f$  is the index-two subgroup  $\pi^{-1}(\mathrm{SL}_2(\mathbb{Z}))$  of  $\mathrm{Aut}(F_2)$  and that the kernel of  $f$  is the center of  $B_4$ .

After the article [9] was published, Etienne Ghys suggested that the action of  $B_4$  on  $F_2$  given by (1.3) might be derived from the fact that a punctured torus is a

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double covering of a disk branched over four points. We checked that this fact indeed led to (1.3).

The present article is a continuation of [9]; it is based on the fact that a punctured surface of arbitrary genus  $g$  can be realized as a ramified double covering of a disk with  $2g + 2$  ramification points. Our main result yields an action of the braid group  $B_{2g+2}$  on  $2g + 2$  strands on the free group  $F_{2g}$  on  $2g$  generators: this action is given by an explicit group homomorphism  $f : B_{2g+2} \rightarrow \text{Aut}(F_{2g})$ , extending (1.3).

The formulas for this homomorphism are given in Section 2; we show in Section 3 how to derive them geometrically. In Section 2.3, in an attempt to define higher analogues of the *special Sturmian monoid*, introduced in [9], we search for automorphisms in  $f(B_{2g+2}) \subset \text{Aut}(F_{2g})$  that preserve the free monoid on  $2g$  generators. It turns out that the situation for  $g \geq 2$  is less satisfactory than for  $g = 1$ .

Finally, in Section 4 we first observe that the image  $f(B_{2g+2})$  maps under the abelianization map  $\pi : \text{Aut}(F_{2g}) \rightarrow \text{GL}_{2g}(\mathbb{Z})$  into the symplectic modular group  $\text{Sp}_{2g}(\mathbb{Z})$ . Concentrating on the case  $g = 2$ , we show that the map  $\pi \circ f : B_6 \rightarrow \text{Sp}_4(\mathbb{Z})$  is surjective and we determine its kernel; we thus obtain a braid-type presentation of  $\text{Sp}_4(\mathbb{Z})$  with generators the standard generators of  $B_6$  and with relations the usual braid relations together with four additional relations.

## 2. THE MAIN RESULT

Let  $g$  be an integer  $\geq 1$  and  $F_{2g}$  be the *free group* on  $2g$  generators  $a_1, \dots, a_g, b_1, \dots, b_g$ .

**2.1. A family of automorphisms of  $F_{2g}$ .** We consider the  $2g + 1$  automorphisms  $u_1, \dots, u_{2g+1}$  of  $F_{2g}$  defined as follows.

- The automorphism  $u_1$  fixes all generators, except  $b_1$  for which we have

$$(2.1) \quad u_1(b_1) = a_1 b_1 .$$

- The automorphism  $u_{2g+1}$  fixes all generators, except  $b_g$  for which

$$(2.2) \quad u_{2g+1}(b_g) = b_g a_g .$$

- For  $i = 1, \dots, g$ , the automorphism  $u_{2i}$  fixes all generators, except  $a_i$  for which

$$(2.3) \quad u_{2i}(a_i) = b_i^{-1} a_i .$$

- For  $i = 1, \dots, g - 1$ , the automorphism  $u_{2i+1}$  fixes all generators, except  $b_i$  and  $b_{i+1}$  for which we have

$$(2.4) \quad u_{2i+1}(b_i) = b_i a_i a_{i+1}^{-1} \quad \text{and} \quad u_{2i+1}(b_{i+1}) = a_{i+1} a_i^{-1} b_{i+1} .$$

When  $g = 1$ , the above  $2g + 1$  automorphisms reduce to three, namely  $u_1, u_2, u_3$ ; these coincide respectively with the automorphisms  $G, D^{-1}, \tilde{G}$  of  $F_2$  defined by (1.1).

**2.2. An action of the braid group  $B_{2g+2}$ .** Let  $B_{2g+2}$  be the braid group on  $2g + 2$  strands with its standard presentation by generators  $\sigma_1, \dots, \sigma_{2g+1}$  satisfying the relations (where  $1 \leq i, j \leq 2g + 1$ )

$$(2.5) \quad \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| > 1$$

and

$$(2.6) \quad \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if } |i - j| = 1 .$$

We now state our main result: it yields an action of  $B_{2g+2}$  on the free group  $F_{2g}$  by group automorphisms.

**Theorem 2.1.** *There is a group homomorphism  $f : B_{2g+2} \rightarrow \text{Aut}(F_{2g})$  such that  $f(\sigma_i) = u_i$  for all  $i = 1, \dots, 2g + 1$ .*

The proof is straightforward: it suffices to check that the automorphisms  $u_1, \dots, u_{2g+1}$  satisfy Relations (2.5) and (2.6). (Formulas (2.1)–(2.4) and Theorem 2.1 were first publicized in [10, Exercise 1.5.2].)

For  $g = 1$ , the homomorphism of Theorem 2.1 coincides with the homomorphism  $f : B_4 \rightarrow \text{Aut}(F_2)$  of [9, Lemma 2.5] defined in the introduction. In *loc. cit.* we showed that its kernel is exactly the center of  $B_4$ . For  $g > 1$  we have the following weaker result.

**Proposition 2.2.** *The kernel of  $f : B_{2g+2} \rightarrow \text{Aut}(F_{2g})$  contains the center of  $B_{2g+2}$ .*

*Proof.* Let  $\delta = \sigma_1 \cdots \sigma_{2g+1}$ . It is well known (see [6, 10]) that the center of  $B_{2g+2}$  is generated by  $\delta^{2g+2}$ . Let  $\tilde{\delta} = f(\delta) = u_1 \cdots u_{2g+1}$  be the corresponding automorphism of  $F_{2g}$ . Using (2.1)–(2.4), it is easy to check that for all  $i = 1, \dots, g$  we have

$$\tilde{\delta}(a_i) = (b_1 \dots b_i)^{-1} \quad \text{and} \quad \tilde{\delta}(b_i) = \begin{cases} a_i a_{i+1}^{-1} & \text{if } i \neq g, \\ a_g & \text{if } i = g. \end{cases}$$

We deduce that

$$\tilde{\delta}^2(a_i) = \begin{cases} a_{i+1} a_1^{-1} & \text{if } i \neq g, \\ a_1^{-1} & \text{if } i = g, \end{cases} \quad \text{and} \quad \tilde{\delta}^2(b_i) = \begin{cases} b_{i+1} & \text{if } i \neq g, \\ (b_1 \dots b_g)^{-1} & \text{if } i = g. \end{cases}$$

From these formulas it is easy to conclude that  $\tilde{\delta}^{2g+2}$  is the identity.  $\square$

**2.3. Preserving the free submonoid  $M_{2g}$  of  $F_{2g}$ .** Let  $M_{2g}$  be the submonoid of  $F_{2g}$  generated by  $a_1, \dots, a_g, b_1, \dots, b_g$ : it is a free monoid. By construction the automorphisms  $u_1$  and  $u_{2g+1}$  preserve  $M_{2g}$ . So do the inverses  $u_{2i}^{-1}$  of the automorphisms  $u_{2i}$  since

$$u_{2i}^{-1}(a_i) = b_i a_i,$$

the other generators being fixed.

Unfortunately, as we can see from (2.4), the automorphisms  $u_{2i+1}$  ( $1 \leq i \leq g-1$ ), which exist only when  $g \geq 2$ , do not preserve the monoid  $M_{2g}$ . Nor do their inverses since

$$u_{2i+1}^{-1}(b_i) = b_i a_{i+1} a_i^{-1} \quad \text{and} \quad u_{2i+1}^{-1}(b_{i+1}) = a_i a_{i+1}^{-1} b_{i+1}.$$

Thus, the action of the higher braid groups  $B_{2g+2}$  on  $F_{2g}$  with  $g \geq 2$  is quite different from the action of  $B_4$  on  $F_2$  when it comes to preserving the monoid  $M_{2g}$ .

Let us consider the case  $g = 1$ . In [9] we observed that the  $M_2$ -preserving automorphisms  $u_1 = G$ ,  $u_2^{-1} = D$ ,  $u_3 = \tilde{G}$  of  $F_2$  are so-called Sturmian morphisms. Together with the Sturmian morphism  $\tilde{D}$ , the morphisms  $G, D, \tilde{G}$  generate the special Sturmian monoid  $\text{St}_0$ , for which we gave a presentation by generators and relations, and which we proved to be isomorphic to the submonoid of  $B_4$  generated by  $\sigma_1, \sigma_2^{-1}, \sigma_3$ , and  $(\sigma_1 \sigma_3^{-1}) \sigma_2^{-1} (\sigma_1 \sigma_3^{-1})^{-1}$ . See [9, Sect. 3] for details.

In the case  $g \geq 2$ , consider the submonoid  $\Omega_{2g}$  of  $\text{Aut}(F_{2g})$  generated by the  $g+2$  elements  $u_1, u_{2g+1}$ , and  $u_{2i}^{-1}$  ( $1 \leq i \leq g$ ). It follows from the observation above that all elements of  $\Omega_{2g}$  preserve the monoid  $M_{2g}$ .

We now express  $\Omega_{2g}$  in terms of the free monoid  $M_2$  on two generators and the free monoid  $M_1$  on one generator, which we identify with the monoid of non-negative integers.

**Proposition 2.3.** *Let  $g \geq 2$ . We have an isomorphism of monoids*

$$\Omega_{2g} \cong M_2 \times (M_1)^{g-2} \times M_2.$$

*Moreover,  $\Omega_{2g}$  is isomorphic to the submonoid of  $B_{2g+2}$  generated by  $\sigma_1, \sigma_{2g+1}$ , and  $\sigma_{2i}^{-1}$  ( $1 \leq i \leq g$ ).*

*Proof.* In view of the commutation relations (2.5) for the automorphisms  $u_i$ , any product of  $u_1, u_2^{-1}, u_4^{-1}, \dots, u_{2g-2}^{-1}, u_{2g}^{-1}, u_{2g+1}$  can be uniquely written as

$$w(u_1, u_2^{-1}) u_4^{-n_2} \cdots u_{2g-2}^{-n_{g-1}} w'(u_{2g}^{-1}, u_{2g+1}),$$

where  $w(u_1, u_2^{-1})$  belongs to the submonoid of  $\text{Aut}(F_{2g})$  generated by  $u_1$  and  $u_2^{-1}$ , the exponents  $n_2, \dots, n_{g-1}$  are non-negative integers, and  $w'(u_{2g}^{-1}, u_{2g+1})$  belongs to the submonoid of  $B_{2g+2}$  generated by  $u_{2g}^{-1}$  and  $u_{2g+1}$ . It remains to show that the submonoid generated by  $u_1, u_2^{-1}$  and the submonoid generated by  $u_{2g}^{-1}, u_{2g+1}$  are both isomorphic to the free monoid  $M_2$ . We give a proof of this claim for the first submonoid (there is a similar proof for the second one). Since  $u_1$  and  $u_2^{-1}$  move only the generators  $a_1$  and  $b_1$ , we may consider them in  $\text{Aut}(F_2)$ . Now let  $w(u_1, u_2^{-1})$  be a non-trivial word in  $u_1, u_2^{-1}$  and consider its image in  $\text{GL}_2(\mathbb{Z})$ ; we have

$$\pi(w(u_1, u_2^{-1})) = w(A, B)$$

where  $A$  and  $B$  are the matrices defined by (1.2). It is well known (and easy to check) that any non-trivial word in  $A, B$  cannot be the identity matrix. Therefore,  $w(u_1, u_2^{-1}) \neq 1$  in  $\text{Aut}(F_2)$ . This proves our claim.

Set  $\iota(u_1) = \sigma_1$ ,  $\iota(u_{2g+1}) = \sigma_{2g+1}$ , and  $\iota(u_{2i}^{-1}) = \sigma_{2i}^{-1}$  for  $i = 1, \dots, g$ . In view of the first assertion and the braid relations (2.5)–(2.6), these formulas define a monoid homomorphism  $\iota : \Omega_{2g} \rightarrow B_{2g+2}$ . Since  $f \circ \iota = \text{id}$  on  $\Omega_{2g}$ , the homomorphism  $\iota$  is injective, which proves the second assertion.  $\square$

### 3. RAMIFIED DOUBLE COVERINGS OF THE DISK

We now explain how we found Formulas (2.1)–(2.4) which define the automorphisms  $u_1, \dots, u_{2g+1}$  of Section 2. The material in this section is standard; we nevertheless give details for the sake of non-topologists.

It is well known that any closed surface  $\Sigma_g$  of genus  $g > 0$  can be realized as a ramified double covering of the sphere  $S^2$  with  $2g + 2$  ramification points. It suffices to embed  $\Sigma_g$  into  $\mathbb{R}^3$  in such a way that it is invariant under the *hyperelliptic involution*, which is the reflexion in a line  $L$  intersecting  $\Sigma_g$  in  $2g + 2$  points as in Figure 1. The quotient of  $\Sigma_g$  by this involution is a sphere equipped with  $2g + 2$  distinguished points, namely the projections of the points of  $\Sigma_g \cap L$ ; these are the ramification points of the double covering.

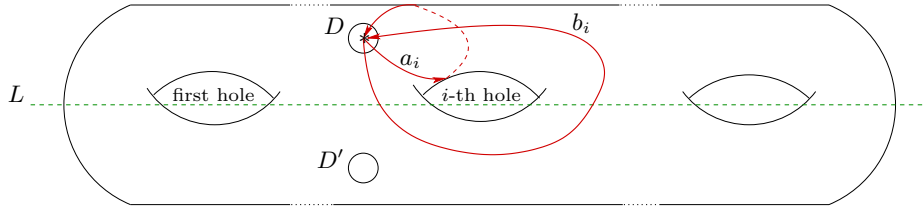


FIGURE 1. A surface  $\Sigma_g$  invariant under the hyperelliptic involution.

From the interior of  $\Sigma_g - \Sigma_g \cap L$  remove a small open disk  $D$  with center  $P$  (represented by  $*$  in the figures), as well as the disk  $D'$  obtained from  $D$  under the reflection in  $L$ . Then  $\Sigma_g^\circ = \Sigma_g - (D \cup D')$  is a ramified double covering of the sphere deprived of a disk, in other words, a double covering of a disk  $D_0$  with  $2g + 2$  ramification points.

As is well known (see [6] or [10, Sect. 1.6]), the braid group  $B_{2g+2}$  is isomorphic to the mapping class group of  $D_0$  consisting of the isotopy classes of orientation-preserving homeomorphisms that fix each point of the boundary of  $D_0$  and permute

the  $2g + 2$  distinguished points. If  $\varphi$  is such an homeomorphism, we pick the lift  $\tilde{\varphi}$  of  $\varphi$  to  $\Sigma_g^\circ$  that fixes  $D$  and  $D'$  pointwise. The correspondence  $\varphi \mapsto \tilde{\varphi}$  induces a homomorphism from  $B_{2g+2}$  to the mapping class group of  $\Sigma_g^\circ$ , hence a homomorphism  $B_{2g+2} \rightarrow \text{Aut}(\Pi)$ , where  $\Pi$  is the fundamental group of  $\Sigma_g^\circ$ , which is a free group. We wish to determine this homomorphism.

It is enough to determine the action of the generators of  $B_{2g+2}$  on a smaller free group, namely the fundamental group of  $\Sigma_g - D'$ , whose elements we represent as loops in  $\Sigma_g - D'$  based at the point  $P$ . This fundamental group is the free group  $F_{2g}$  generated by the loops  $a_1, \dots, a_g, b_1, \dots, b_g$  as depicted in Figure 1.

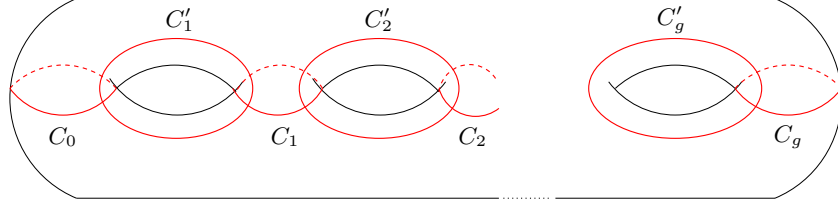


FIGURE 2. The curves  $C_i$  and  $C'_i$ .

Consider the curves  $C_0, C_1, \dots, C_g, C'_1, \dots, C'_g$  of Figure 2. It is easy to check that a lift  $\tilde{\sigma}_1$  of the homeomorphism of  $D_0$  representing the generator  $\sigma_1$  of  $B_{2g+2}$  is the Dehn twist  $T_0$  along the curve  $C_0$  (for a definition of Dehn twists, see [10, Sect. 3.2.4]). The action of  $T_0$  on the generators of the fundamental group of  $\Sigma_g - D'$  is easy to compute; clearly it leaves the generators  $a_1, \dots, a_g$  as well as  $b_2, \dots, b_g$  unchanged. On  $b_1$  it acts as in Figure 3, which leads to Formula (2.1).

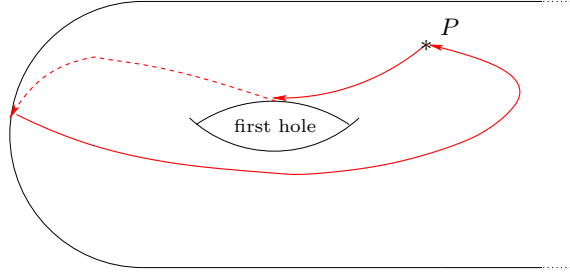


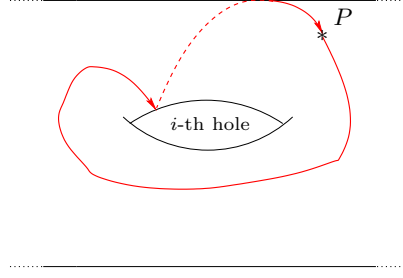
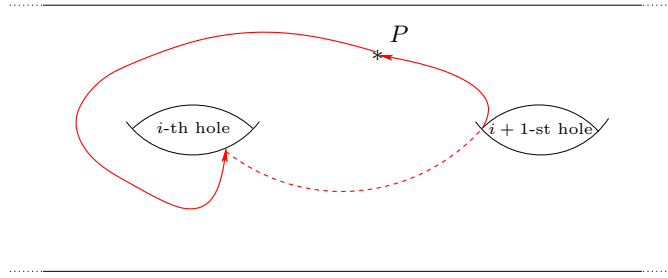
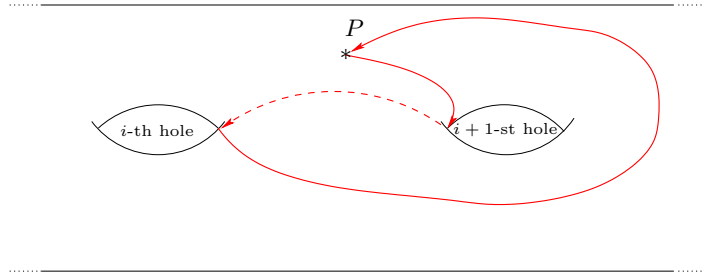
FIGURE 3. Action of the Dehn twist  $T_0$  on  $b_1$ .

Similarly, a lift  $\tilde{\sigma}_{2g+1}$  of the homeomorphism representing the last generator  $\sigma_{2g+1}$  of  $B_{2g+2}$  is the Dehn twist  $T_g$  along the curve  $C_g$ . This twist fixes all generators of the fundamental group except  $b_g$ . The reader is encouraged to draw the corresponding figure and derive (2.2) from it.

A lift  $\tilde{\sigma}_{2i}$  of the homeomorphism representing the generator  $\sigma_{2i}$  ( $1 \leq i \leq g$ ) is the Dehn twist  $T'_i$  along the curve  $C'_i$ . This twist affects only the generator  $a_i$ , on which it acts as in Figure 4; we thus obtain (2.3).

Finally, when  $1 \leq i \leq g-1$ , a lift  $\tilde{\sigma}_{2i+1}$  of the homeomorphism representing  $\sigma_{2i+1}$  is the Dehn twist  $T_i$  along the curve  $C_i$ . This twist fixes all generators of the fundamental group except  $b_i$  and  $b_{i+1}$ ; it acts on  $b_i$  as in Figure 5 and on  $b_{i+1}$  as in Figure 6. This yields Formula (2.4).

The above computation appeared in [4] with different notation; the correspondence between that paper and ours is given by  $s_i = b_i^{-1}$ ,  $t_i = a_i^{-1}$ ,  $h_{C_1} = T_0^{-1}$ ,  $h_{U_i} = T'_i{}^{-1}$ , and  $h_{Z_i} = T_i^{-1}$  (see also [8, Sect. 4] and [3, Sect. 2]). A proof of Proposition 2.2 can also be derived from Relation (10) in [4].

FIGURE 4. Action of the Dehn twist  $T'_i$  on  $a_i$  ( $1 \leq i \leq g$ ).FIGURE 5. Action of the Dehn twist  $T_i$  on  $b_i$ .FIGURE 6. Action of the Dehn twist  $T_i$  on  $b_{i+1}$ .

#### 4. THE SYMPLECTIC MODULAR GROUP

In this section we first observe that, after abelianizing our action, we obtain a symplectic action of the braid group  $B_{2g+2}$  on the free abelian group  $\mathbb{Z}^{2g}$ . In the second part of the section we elaborate on the case  $g = 2$  and obtain a braid-type presentation of  $\mathrm{Sp}_4(\mathbb{Z})$ .

**4.1. Symplectic automorphisms.** Let  $g \geq 1$  be a positive integer. Pick a basis  $(\bar{a}_1, \dots, \bar{a}_g, \bar{b}_1, \dots, \bar{b}_g)$  of  $\mathbb{Z}^{2g}$ . Using this basis, we identify the group  $\mathrm{Aut}(\mathbb{Z}^{2g})$  of automorphisms of  $\mathbb{Z}^{2g}$  with the general linear group  $\mathrm{GL}_{2g}(\mathbb{Z})$ .

We equip  $\mathbb{Z}^{2g}$  with the standard alternating bilinear form  $\langle \cdot, \cdot \rangle$  which vanishes on all pairs of basis elements, except the following ones:

$$\langle \bar{a}_i, \bar{b}_i \rangle = -\langle \bar{b}_i, \bar{a}_i \rangle = 1 \quad (i = 1, \dots, g).$$

The *symplectic modular group*  $\mathrm{Sp}_{2g}(\mathbb{Z})$  (formerly called Siegel's modular group) is the group of automorphisms of  $\mathbb{Z}^{2g}$  preserving this alternating form; it can be

described as the group of matrices  $M \in \mathrm{GL}_{2g}(\mathbb{Z})$  such that

$$(4.1) \quad M^T \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix},$$

where  $M^T$  is the transpose of  $M$  and  $I_g$  is the identity matrix of size  $g$ .

Consider the composition

$$\bar{f} : B_{2g+2} \rightarrow \mathrm{GL}_{2g}(\mathbb{Z})$$

of the homomorphism  $f : B_{2g+2} \rightarrow \mathrm{Aut}(F_{2g})$  of Section 2 with the natural surjection  $\pi : \mathrm{Aut}(F_{2g}) \rightarrow \mathrm{GL}_{2g}(\mathbb{Z})$ .

**Proposition 4.1.** *We have  $\bar{f}(B_{2g+2}) \subset \mathrm{Sp}_{2g}(\mathbb{Z})$ .*

*Proof.* It is enough to check that the image  $\pi(u_i)$  in  $\mathrm{GL}_{2g}(\mathbb{Z})$  of each automorphism  $u_i$  belongs to  $\mathrm{Sp}_{2g}(\mathbb{Z})$ . Now the automorphisms  $u_i$  are induced by elements of a mapping class group which are well known to induce symplectic linear maps (see [12, Sect. 5.8]). Alternatively, one checks that each matrix  $\pi(u_i)$  satisfies Relation (4.1).  $\square$

When  $g = 1$ , the symplectic group  $\mathrm{Sp}_2(\mathbb{Z})$  identifies naturally with the modular group  $\mathrm{SL}_2(\mathbb{Z})$ . We proved in [9] that the image  $\bar{f}(B_4)$  is the entire group  $\mathrm{SL}_2(\mathbb{Z})$ .

**Remark 4.2.** Taking an adequate specialization of the Burau representation, Magnus and Peloso [13] also constructed a homomorphism  $B_{2g+2} \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$ , for which they showed that it is surjective if and only if  $g = 1$  or  $g = 2$ . We do not know if their symplectic representation is related to ours.

**4.2. The case  $g = 2$ .** We now consider the homomorphism  $\bar{f} : B_{2g+2} \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$  in the special case  $g = 2$ .

**Theorem 4.3.** *The homomorphism  $\bar{f} : B_6 \rightarrow \mathrm{Sp}_4(\mathbb{Z})$  is surjective and its kernel is the normal subgroup of  $B_6$  generated by  $\Delta^2$ ,  $\alpha^2$ ,  $\alpha\beta$ ,  $(\alpha\gamma)^2$ , where*

$$\begin{aligned} \Delta &= \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1, \\ \alpha &= (\sigma_4\sigma_5)^3, \quad \beta = \sigma_3^{-1}(\sigma_1\sigma_2)^3\sigma_3, \quad \gamma = \sigma_1\sigma_3^{-1}\sigma_5. \end{aligned}$$

As a consequence, we obtain the following braid-type presentation of  $\mathrm{Sp}_4(\mathbb{Z})$ .

**Corollary 4.4.** *The symplectic group  $\mathrm{Sp}_4(\mathbb{Z})$  has a presentation with generators  $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$  and 14 relations consisting of Relations (2.5)–(2.6) and of the four relations*

$$\begin{aligned} (\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1)^2 &= 1, \\ (\sigma_4\sigma_5)^6 &= 1, \quad (\sigma_1\sigma_2)^3 = \sigma_3(\sigma_4\sigma_5)^3\sigma_3^{-1}, \\ (\sigma_1\sigma_3^{-1}\sigma_5)^{-1} &= (\sigma_4\sigma_5)^3(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_4\sigma_5)^{-3}. \end{aligned}$$

Previously Behr [1] gave a quite different presentation of this group, with six generators and 18 relations (see below); Bender [2] improved Behr's presentation by reducing it to one with two generators and 8 relations (see also [5]).

*Proof of Theorem 4.3.* The generators in Behr's presentation [1] of  $\mathrm{Sp}_4(\mathbb{Z})$  are the following symplectic matrices:

$$x_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x_{\alpha+\beta} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x_{2\alpha+\beta} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$x_\alpha = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad w_\alpha = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad w_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The images under  $\bar{f}$  of the elements  $\sigma_1, \dots, \sigma_5$ , and  $\Delta$  of  $B_6$  are given by

$$\begin{aligned} M_1 = \bar{f}(\sigma_1) &= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & M_2 = \bar{f}(\sigma_2) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ M_3 = \bar{f}(\sigma_3) &= \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & M_4 = \bar{f}(\sigma_4) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \\ M_5 = \bar{f}(\sigma_5) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & M_\Delta = \bar{f}(\Delta) &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \end{aligned}$$

One checks that

$$(4.2) \quad x_\beta = M_5, \quad x_{\alpha+\beta} = M_1 M_3^{-1} M_5, \quad x_{2\alpha+\beta} = M_1,$$

$$(4.3) \quad x_\alpha = M_5^{-1} M_4^{-1} M_1^{-1} M_3 M_5^{-1} M_4 M_5,$$

$$(4.4) \quad w_\alpha = (M_4 M_5)^3 M_\Delta, \quad w_\beta = (M_4 M_5 M_4)^{-1}.$$

This shows that Behr's generators all belong to the image of the homomorphism  $\bar{f}$ , which proves the surjectivity of  $\bar{f}$ .

Each among the 18 relations in Behr's presentation yields a generator of the kernel of  $\bar{f}$  as follows: we first write each relation as an equality  $r_i = 1$ , where  $r_i$  is a word in Behr's generators and their inverses ( $1 \leq i \leq 18$ ); then in view of (4.2)–(4.4), we replace in  $r_i$  each generator by the following corresponding lift in  $B_6$ :

$$\begin{aligned} x_\beta &\rightsquigarrow \sigma_5, & x_{\alpha+\beta} &\rightsquigarrow \sigma_1 \sigma_3^{-1} \sigma_5, & x_{2\alpha+\beta} &\rightsquigarrow \sigma_1, \\ x_\alpha &\rightsquigarrow \sigma_5^{-1} \sigma_4^{-1} \sigma_1^{-1} \sigma_3 \sigma_5^{-1} \sigma_4 \sigma_5, \\ w_\alpha &\rightsquigarrow (\sigma_4 \sigma_5)^3 \Delta, & w_\beta &\rightsquigarrow (\sigma_4 \sigma_5 \sigma_4)^{-1}. \end{aligned}$$

In this way, from each relation  $r_i = 1$  we obtain an element  $\gamma_i \in B_6$ , which belongs to the kernel of  $\bar{f}$ . It follows from [1, Satz] that the 18 elements  $\gamma_1, \dots, \gamma_{18}$  generate a subgroup whose normal closure is the kernel of  $\bar{f}$ .

It is easy to check that  $\gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_8, \gamma_9, \gamma_{11}, \gamma_{12}, \gamma_{15}, \gamma_{16}$ , and  $\gamma_{18}$  are all equal to the trivial element of  $B_6$ . Therefore, the kernel of  $\bar{f}$  is the normal closure of the subgroup generated by the remaining elements  $\gamma_1, \gamma_2, \gamma_7, \gamma_{10}, \gamma_{13}, \gamma_{14}, \gamma_{17}$ .

Let us now handle these seven elements one by one. Behr's Relation (1) is equivalent to  $r_1 = 1$ , where  $r_1 = x_{2\alpha+\beta}^{-1} x_{\alpha+\beta}^{-1} x_\alpha x_\beta x_\alpha^{-1} x_\beta^{-1}$ . We thus have

$$(4.5) \quad \gamma_1 = \sigma_1^{-1} (\sigma_5^{-1} \sigma_3 \sigma_1^{-1}) (\sigma_5^{-1} \sigma_4^{-1} \sigma_1^{-1} \sigma_3 \sigma_5^{-1} \sigma_4 \sigma_5) \sigma_5 (\sigma_5^{-1} \sigma_4^{-1} \sigma_1 \sigma_3^{-1} \sigma_5 \sigma_4 \sigma_5) \sigma_5^{-1}.$$

Relation (2) in [1] is equivalent to  $r_2 = 1$ , where  $r_2 = x_{2\alpha+\beta}^{-2} x_\alpha x_{\alpha+\beta} x_\alpha^{-1} x_{\alpha+\beta}^{-1}$ . Therefore,

$$(4.6) \quad \gamma_2 = \sigma_1^{-2} (\sigma_5^{-1} \sigma_4^{-1} \sigma_1^{-1} \sigma_3 \sigma_5^{-1} \sigma_4 \sigma_5) \times (\sigma_1 \sigma_3^{-1} \sigma_5) (\sigma_5^{-1} \sigma_4^{-1} \sigma_1 \sigma_3^{-1} \sigma_5 \sigma_4 \sigma_5) (\sigma_1^{-1} \sigma_3 \sigma_5^{-1}).$$



Using the braid applet of [7], we deduce from (4.5) and (4.6) that

$$(4.7) \quad \gamma_1 = \sigma_3 \gamma_2 \sigma_3^{-1}.$$

Behr's Relation (7) is equivalent to  $r_7 = 1$  with  $r_7 = w_\alpha w_\beta^2 w_\alpha w_\beta^{-2}$ . We thus have

$$\begin{aligned} \gamma_7 &= (\sigma_4 \sigma_5)^3 \Delta (\sigma_4 \sigma_5 \sigma_4)^{-2} (\sigma_4 \sigma_5)^3 \Delta (\sigma_4 \sigma_5 \sigma_4)^2 \\ &= (\sigma_4 \sigma_5)^3 \Delta (\sigma_4 \sigma_5 \sigma_4)^{-2} (\sigma_4 \sigma_5 \sigma_4)^2 \Delta (\sigma_4 \sigma_5)^3 \\ &= (\sigma_4 \sigma_5)^3 \Delta^2 (\sigma_4 \sigma_5)^3. \end{aligned}$$

Since  $\Delta^2$  is central, as is well known (see [6, 10]), we obtain

$$(4.8) \quad \gamma_7 = (\sigma_4 \sigma_5)^6 \Delta^2.$$

Behr's Relation (10), which is equivalent to  $w_\beta^{-4} = 1$ , yields

$$(4.9) \quad \gamma_{10} = (\sigma_4 \sigma_5 \sigma_4)^4 = (\sigma_4 \sigma_5)^6.$$

Relation (13) in [1] is equivalent to  $r_{13} = 1$ , where  $r_{13} = w_\alpha x_{\alpha+\beta} w_\alpha^{-1} x_{\alpha+\beta}$ . Therefore,

$$\gamma_{13} = (\sigma_4 \sigma_5)^3 \Delta (\sigma_1 \sigma_3^{-1} \sigma_5) \Delta^{-1} (\sigma_4 \sigma_5)^{-3} (\sigma_1 \sigma_3^{-1} \sigma_5).$$

In view of the relations  $\Delta \sigma_i = \sigma_{6-i} \Delta$  ( $i = 1, \dots, 5$ ), we obtain

$$(4.10) \quad \gamma_{13} = (\sigma_4 \sigma_5)^3 (\sigma_1 \sigma_3^{-1} \sigma_5) (\sigma_4 \sigma_5)^{-3} (\sigma_1 \sigma_3^{-1} \sigma_5).$$

Using again the braid applet [7], we find

$$(4.11) \quad \gamma_{13} = \gamma_2^{-1}.$$

Relation (14) in [1] is equivalent to  $r_{14} = 1$ , where  $r_{14} = x_\alpha w_\beta^{-1} x_{\alpha+\beta}^{-1} w_\beta$ . Therefore,

$$\gamma_{14} = (\sigma_5^{-1} \sigma_4^{-1} \sigma_1^{-1} \sigma_3 \sigma_5^{-1} \sigma_4 \sigma_5) (\sigma_4 \sigma_5 \sigma_4) (\sigma_1^{-1} \sigma_3 \sigma_5^{-1}) (\sigma_4^{-1} \sigma_5^{-1} \sigma_4^{-1}).$$

We similarly find

$$(4.12) \quad \gamma_{14} = \gamma_2 = \gamma_{13}^{-1}.$$

Relation (17) in [1] is equivalent to  $r_{17} = 1$  with  $r_{17} = w_\alpha x_\alpha w_\alpha^{-1} x_\alpha w_\alpha x_\alpha$ . Thus,

$$\begin{aligned} \gamma_{17} &= (\sigma_4 \sigma_5)^3 \Delta (\sigma_5^{-1} \sigma_4^{-1} \sigma_1^{-1} \sigma_3 \sigma_5^{-1} \sigma_4 \sigma_5) \Delta^{-1} (\sigma_4 \sigma_5)^{-3} \times \\ &\quad \times (\sigma_5^{-1} \sigma_4^{-1} \sigma_1^{-1} \sigma_3 \sigma_5^{-1} \sigma_4 \sigma_5) (\sigma_4 \sigma_5)^3 \Delta (\sigma_5^{-1} \sigma_4^{-1} \sigma_1^{-1} \sigma_3 \sigma_5^{-1} \sigma_4 \sigma_5). \end{aligned}$$

Letting both  $\Delta$  jump to the right and letting the leftmost one merge with  $\Delta^{-1}$ , we obtain

$$(4.13) \quad \begin{aligned} \gamma_{17} &= (\sigma_4 \sigma_5)^3 (\sigma_1 \sigma_2 \sigma_1^{-1} \sigma_3 \sigma_5^{-1} \sigma_2^{-1} \sigma_1^{-1}) (\sigma_4 \sigma_5)^{-3} \times \\ &\quad \times (\sigma_5^{-1} \sigma_4^{-1} \sigma_1^{-1} \sigma_3 \sigma_5^{-1} \sigma_4 \sigma_5) (\sigma_4 \sigma_5)^3 (\sigma_1 \sigma_2 \sigma_1^{-1} \sigma_3 \sigma_5^{-1} \sigma_2^{-1} \sigma_1^{-1}) \Delta. \end{aligned}$$

It follows from (4.7), (4.11), (4.12) that the kernel of  $\bar{f}$  is the normal closure of the subgroup generated by  $\gamma_7$ ,  $\gamma_{10}$ ,  $\gamma_{13}$ , and  $\gamma_{17}$ . Now, by (4.8)–(4.10) we have  $\gamma_7 = \alpha^2 \Delta^2$ ,  $\gamma_{10} = \alpha^2$ , and

$$\gamma_{13} = \alpha \gamma \alpha^{-1} \gamma = (\alpha \gamma)^2 \gamma^{-1} (\alpha^2)^{-1} \gamma,$$

where  $\Delta$ ,  $\alpha$ ,  $\gamma$  have been defined in the statement of the theorem. Hence, the kernel of  $\bar{f}$  is generated as a normal subgroup by  $\Delta^2$ ,  $\alpha^2$ ,  $(\alpha \gamma)^2$ , and  $\gamma_{17}$ . To complete the proof of the theorem, we consider the normal subgroup  $N$  of  $B_6$  generated by  $\Delta^2$ ,  $\alpha^2$ , and  $(\alpha \gamma)^2$ , and we show that  $\gamma_{17}$  is conjugate to  $\alpha \beta$  modulo  $N$ .

Let us now prove this. From (4.13) we first derive

$$\gamma_{17} = \sigma_1 \sigma_2 \alpha \gamma^{-1} \sigma_2^{-1} \sigma_1^{-1} \alpha^{-1} \sigma_5^{-1} \sigma_4^{-1} \gamma^{-1} \sigma_4 \sigma_5 \alpha \sigma_1 \sigma_2 \gamma^{-1} \sigma_2^{-1} \sigma_1^{-1} \Delta.$$

Now,  $(\alpha\gamma)^2 \equiv 1$  modulo  $N$ . Hence,  $\gamma^{-1} \equiv \alpha\gamma\alpha$ . Therefore, in view of this and of  $\alpha^2 \equiv 1$ , we obtain

$$\begin{aligned}\gamma_{17} &\equiv \sigma_1\sigma_2\alpha^2\gamma\alpha\sigma_2^{-1}\sigma_1^{-1}\alpha^{-1}\sigma_5^{-1}\sigma_4^{-1}\alpha\gamma\alpha\sigma_4\sigma_5\alpha\sigma_1\sigma_2\alpha\gamma\alpha^{-1}\alpha^2\sigma_2^{-1}\sigma_1^{-1}\Delta \\ &\equiv \sigma_1\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_5\sigma_5^{-1}\sigma_4^{-1}\alpha\gamma\sigma_4\sigma_5\sigma_1\sigma_2\alpha\gamma\alpha^{-1}\sigma_2^{-1}\sigma_1^{-1}\Delta \\ &\equiv \sigma_1\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_4^{-1}\alpha\sigma_1\sigma_3^{-1}\sigma_5\sigma_4\sigma_5\sigma_1\sigma_2\alpha\gamma\alpha^{-1}\sigma_2^{-1}\sigma_1^{-1}\Delta \\ &\equiv (\sigma_1\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_4^{-1})\alpha\sigma_3^{-1}\sigma_1\sigma_2\sigma_5\sigma_4\sigma_5\alpha\gamma(\alpha^{-1}\sigma_2^{-1}\sigma_1^{-1}\Delta)\end{aligned}$$

modulo  $N$ . Using the applet [7], we obtain

$$\alpha^{-1}\sigma_2^{-1}\sigma_1^{-1}\Delta = \sigma_3\sigma_4\sigma_5\sigma_2\sigma_1\sigma_2\sigma_3(\sigma_1\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_4^{-1})^{-1}.$$

Therefore,  $\gamma_{17} \equiv (\sigma_1\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_4^{-1})\gamma'(\sigma_1\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_4^{-1})^{-1}$ , where

$$\begin{aligned}\gamma' &= \alpha\sigma_3^{-1}\sigma_1\sigma_2\sigma_5\sigma_4\sigma_5\alpha\gamma\sigma_3\sigma_4\sigma_5\sigma_2\sigma_1\sigma_2\sigma_3 \\ &= \alpha\sigma_3^{-1}(\sigma_1\sigma_2\sigma_5\sigma_4\sigma_5\alpha\sigma_5\sigma_1\sigma_3^{-1}\sigma_3\sigma_4\sigma_5\sigma_2\sigma_1\sigma_2)\sigma_3 \\ &= \alpha\sigma_3^{-1}((\sigma_1\sigma_2)(\sigma_5\sigma_4\sigma_5\alpha\sigma_5\sigma_4\sigma_5)(\sigma_1\sigma_2)^2)\sigma_3 \\ &= \alpha\sigma_3^{-1}((\sigma_1\sigma_2)\alpha^2(\sigma_1\sigma_2)^2)\sigma_3 \\ &\equiv \alpha\sigma_3^{-1}(\sigma_1\sigma_2)^3\sigma_3 = \alpha\beta.\end{aligned}$$

This completes the proof of Theorem 4.3.  $\square$

**Remark 4.5.** We may wonder whether the four elements  $\Delta^2$ ,  $\alpha^2$ ,  $\alpha\beta$ ,  $(\alpha\gamma)^2$  of the kernel of  $\bar{f}$  already belong to the kernel of  $f$ . This holds true for the central element  $\Delta^2$ : indeed,  $f(\Delta^2) = \text{id}$  by Proposition 2.2. The other three elements map under  $f$  to non-inner automorphisms of  $F_4$ . For instance, the automorphism  $f(\alpha^2)$  is the following: it fixes  $a_1$  and  $b_1$ , and on  $a_2$  and  $b_2$  we have

$$a_2 \mapsto (a_2^{-1}b_2a_2b_2^{-1})a_2(a_2^{-1}b_2a_2b_2^{-1})^{-1} \quad \text{and} \quad b_2 \mapsto (a_2^{-1}b_2a_2)b_2(a_2^{-1}b_2a_2)^{-1}.$$

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#### REFERENCES

- [1] H. Behr, Eine endliche Präsentation der symplektischen Gruppe  $\text{Sp}_4(\mathbb{Z})$ , *Math. Z.* **141** (1975), 47–56.
- [2] P. Bender, Eine Präsentation der symplektischen Gruppe  $\text{Sp}_4(\mathbb{Z})$  mit 2 Erzeugenden und 8 definierenden Relationen, *J. Algebra* **65** (1980), 328–331.
- [3] P. Bergau, J. Mennicke, Über topologische Abbildungen der Brezelfläche vom Geschlecht 2, *Math. Z.* **74** (1960), 414–435.
- [4] J. S. Birman, Automorphisms of the fundamental group of a closed, orientable 2-manifold, *Proc. Amer. Math. Soc.* **21** (1969) 351–354.
- [5] J. S. Birman, On Siegel’s modular group, *Math. Ann.* **191** (1971), 59–68.
- [6] J. S. Birman, *Braids, Links and Mapping Class Groups*, Annals of Math. Studies, No. 82 (Princeton University Press, Princeton, 1975).
- [7] P. Dehornoy, J. Fromentin, A braid applet, [www.math.unicaen.fr/~tressapp/index.html](http://www.math.unicaen.fr/~tressapp/index.html).
- [8] L. Goeritz, Die Abbildungen der Brezelfläche und der Vollbrezel vom Geschlecht 2, *Abh. Math. Sem. Univ. Hamburg* **9** (1933), 244–259.
- [9] C. Kassel, C. Reutenauer, Sturmian morphisms, the braid group  $B_4$ , Christoffel words and bases of  $F_2$ , *Ann. Mat. Pura Appl. (4)* **186** (2007) 317–339.
- [10] C. Kassel, V. Turaev, *Braid groups*, Grad. Texts in Math., Vol. 247 (Springer, New York, 2008).
- [11] M. Lothaire, *Algebraic combinatorics on words*, 45–110 (Cambridge University Press, Cambridge, 2002).
- [12] W. Magnus, A. Karrass, D. Solitar, *Combinatorial group theory* (John Wiley and Sons, New York, 1966).

- [13] W. Magnus, A. Peluso, On a theorem of V. I. Arnol'd, *Comm. Pure Appl. Math.* **22** (1969) 683–692.

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